A note on the analogy between superfluids and cosmology

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A new analogy between superfluid systems and cosmology is here presented, which relies strongly on the following ingredient: the back-reaction of the vacuum to the quanta of sound waves. We show how the presence of thermal phonons, the excitations above the quantum vacuum for $T > 0$, enable us to deduce an hydrodynamical equation formally similar to the one obtained for a perfect fluid in a Universe obeying the Friedmann-Robertson-Walker metric.

I. INTRODUCTION

Different condensed matter systems, such as acoustics in flowing fluids, light in moving dielectrics or quasiparticles in moving superfluids, can be shown to reproduce some aspects of General Relativity (GR) and cosmology[1,2]. They can be conceived as laboratory toy models in order to make experimentally accessible some features of quantum field theory on curved-space. The starting point of such an exciting and fruitful research area can be found in the celebrated *acoustic black hole* by Unruh[3]. Here the analogy between the motion of sound waves in a convergent fluid flow and massless spin-zero particles exposed to a black hole was first outlined. Since then, the search for an emergent space-time has been extended to various media, such as electromagnetic waveguides[4], superfluid helium[1] and Bose-Einstein condensates[5]. Emergent space-time and gravity effects in superfluids are of particular interest. Indeed the extremely low temperatures experimentally accessible allow in principle the detection of tiny quantum effects, such as Hawking radiation, particle production and quantum back-reaction[6,7]. Bose-Einstein condensates made of cold atoms in optical lattices are very promising because of the high degree of experimental control[8,9]. Indeed such systems have been proposed to mimic an expanding Friedmann, Robertson, Walker (FRW) universe[10], where the behavior of quantum modes has been reproduced by manipulating the speed of sound through external fields via Feshbach resonance techniques[11].

On the other hand the simulation of some gravity effects in condensed matter systems leads to new insights into the deep connection between quantum hydrodynamics and quantum gravity[6]. In this context one of the main open problems is the correct treatment of quantum fluctuations, always present on the top of the classical background which describes the macroscopic behavior of superfluids. This quantum back-reaction is also related to fundamental issues such as the Big Bang singularity and the cosmological constant.

In this letter we take a step forward in the introduction of a new analogy between superfluid systems and gravity, which relies strongly on the analysis of the back-reaction of the vacuum to the quanta of sound waves. In particular, we show how the presence of thermal phonons, the excitations above the quantum vacuum for $T > 0$, allows us to justify an hydrodynamical equation formally similar to the one obtained for a perfect fluid in a Universe obeying the FRW metric[10]. The letter is organized as follows. In Section 2 we briefly recall the derivation of the Friedmann fluid equation within the FRW cosmological model. In Section 3 we show that the non-zero temperature and the back-reaction of quanta of sound waves onto the quantum vacuum allow us to derive a cosmological-like equation. Finally, in Section 4 some comments and outlooks of this work are given.

II. THE COSMOLOGICAL FLUID EQUATION

In this Section we briefly review how to derive the Friedmann fluid equation within the FRW cosmological model[10].

As a starting point our Universe is assumed homogeneous and isotropic and this hypothesis is perfectly compatible with observations on large scales of length $\sim 4000 Mps$. Such an Universe is described by means of the FRW metric

tensor:

$$[g_{\mu\nu}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{a^2(t)}{1-kr^2} & 0 & 0 \\ 0 & 0 & -a^2(t)r^2 & 0 \\ 0 & 0 & 0 & -a^2(t)r^2\sin^2\vartheta \end{pmatrix},$$

where $k = +1,\, 0,\, -1$ is the sign of the curvature and $a(t)$ is the expansion factor (case of non-stationary Universe), which is a function of the time alone in order to have a Universe homogeneous and isotropic.

The dynamical link between the matter content of the Universe and the metric tensor is codified in the Einstein's equations:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R, \tag{II.1}$$

with $G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$; here c is the light speed, G is the gravitational constant, $T_{\mu\nu}$ is the energy-momentum tensor, $R_{\mu\nu}$ is the Ricci curvature tensor, R is the scalar curvature and $g_{\mu\nu}$ is the metric tensor. Einstein's equations tell us that *the presence of matter bends space-time*. In the particular case in which there is absence of matter we have $T_{\mu\nu} = 0$. For a perfect fluid, that is a fluid which has no viscosity or heat flow, the energy-momentum tensor can be written as:

$$T_{\mu\nu} = \left(\frac{p}{c^2} + \rho\right)u_\mu u_\nu - pg_{\mu\nu},$$

where p is the pressure, ρ is the density and u the fluid velocity. In a co-moving system, that is a system at rest with respect to the cosmic fluid, the fluid velocity is $u = (c, 0, 0, 0)$ and then:

$$T_{00} = \rho c^2, \qquad T_{11} = \frac{p\,a^2(t)}{1 - k\,r^2}, \qquad T_{22} = p\,r^2 a^2(t), \qquad T_{33} = p\,r^2 a^2(t)\sin^2\vartheta. \tag{II.2}$$

Here T_{00} is the energy density while T_{11}, T_{22} and T_{33} give rise to the pressure. From II.2 it follows that:

$$[T^\mu{}_\nu] = \begin{pmatrix} c^2\rho & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{pmatrix}.$$

Let us now obtain the Friedmann fluid equation. In order to pursue this task, let us remember that the Einstein's field equation must satisfy the *Bianchi identity*

$$D_\nu G^{\mu\nu} \equiv D_\nu T^{\mu\nu} \equiv T^{\mu\nu}{}_{;\nu} = 0,$$

where D_ν is the *covariant derivative* and the rising of the two indices in $G_{\mu\nu}$ and in $T_{\mu\nu}$ is obtained by means of two applications of the metric tensor $g^{\mu\nu} = g_{\mu\nu}^{-1}$. So, we get:

$$T^{\mu\nu}{}_{;\nu} = \frac{\partial T^{\mu\nu}}{\partial x^\nu} + \Gamma^\mu{}_{\nu\sigma}T^{\sigma\nu} + \Gamma^\nu{}_{\nu\sigma}T^{\mu\sigma} = 0. \tag{II.3}$$

The *connection coefficients* or *Christoffel symbols*, which are not tensors, $\Gamma^\mu{}_{\nu\sigma}$ are given by

$$\Gamma^\mu{}_{\nu\sigma} = \frac{1}{2}g^{\mu\lambda}\left(\frac{\partial g_{\lambda\nu}}{\partial x^\sigma} + \frac{\partial g_{\lambda\sigma}}{\partial x^\nu} - \frac{\partial g_{\nu\sigma}}{\partial x^\lambda}\right).$$

In order to understand their physical meaning, let us analyze the *parallel transport* of an arbitrary contravariant vector A^α in a curved space-time, which describes the Universe in General Relativity. In particular if its value at a point x^α is A^α, then at the neighboring point $x^\alpha + dx^\alpha$ it is equal to $A^\alpha + dA^\alpha$. Now let the vector A^α perform an infinitesimal parallel displacement to the point $x^\alpha + dx^\alpha$. As a result of such an operation, the change in the vector A^α is denoted by δA^α. Then the difference DA^α between the two vectors which are now located at the same point is:

$$DA^\alpha = dA^\alpha - \delta A^\alpha. \tag{II.4}$$

It is possible to show that[12] $\delta A^\alpha = -\Gamma^\alpha{}_{\mu\nu}A^\mu dx^\nu$. For a galilean coordinate system the following relation holds: $\Gamma^\alpha{}_{\mu\nu} = 0$. We explicitly note that the equation (II.4) gives us the definition of the covariant derivative of a contravariant vector once we divide both sides by dx^ν. Furthermore the generalization to a two indices tensor is provided by Eq. (II.3).

Starting from Eq. (II.3), after some manipulations we finally obtain the fluid equation:

$$\dot\rho + 3\frac{\dot a}{a}\left(\rho + \frac{p}{c^2}\right) = 0 \qquad \Rightarrow \qquad \frac{d\rho}{da} = -\frac{3}{a}\left(\rho + \frac{p}{c^2}\right). \tag{II.5}$$

Now we wonder if and under which hypothesis it is possible to obtain a similar equation in a quantum hydrodynamics context. We address this point in the following Section.

III.   ONE MORE SIMILITUDE

In this Section we perform a step forward in the introduction of a new analogy between superfluids and cosmology by working in a quantum hydrodynamics context. Within such a context, we need to carefully include in our analysis the effect of the back-reaction of the vacuum to the quanta of sound waves at non-zero temperature which leads to a depletion of the mass density ρ. In particular, we are going to show how the presence of thermal phonons, the excitations above the quantum vacuum for $T > 0$, allows us to justify an hydrodynamical equation formally similar to the one obtained for a perfect fluid in a Universe obeying the FRW metric[10].

In order to clarify the physical picture behind our derivation let us briefly recall the main concepts on which Landau's formulation of quantum hydrodynamics strongly relies[13]. The starting point is the quantum Hamiltonian

$$H\left(\widehat{\rho}, \widehat{\mathbf{v}}\right) = \int d^3x \left(\frac{1}{2}\widehat{\mathbf{v}}\widehat{\rho}\widehat{\mathbf{v}} + \epsilon\left(\widehat{\rho}\right) - \mu\widehat{\rho} \right), \tag{III.6}$$

where ϵ is the energy density of the liquid, μ is the chemical potential and the quantum operators $\widehat{\mathbf{v}}$ (velocity field operator) and $\widehat{\rho}$ (mass density operator) satisfy the following commutation rules:

$$[\widehat{\rho}\left(\mathbf{r}_1\right), \widehat{\rho}\left(\mathbf{r}_2\right)] = 0, \tag{III.7}$$

$$[\widehat{\mathbf{v}}\left(\mathbf{r}_1\right), \widehat{\rho}\left(\mathbf{r}_2\right)] = \frac{\hbar}{i}\nabla\delta\left(\mathbf{r}_1 - \mathbf{r}_2\right), \tag{III.8}$$

$$[\widehat{v}_i\left(\mathbf{r}_1\right), \widehat{v}_j\left(\mathbf{r}_2\right)] = \frac{\hbar}{i\widehat{\rho}}\varepsilon_{ijk}\left(\nabla \times \widehat{\mathbf{v}}\right)_k \delta\left(\mathbf{r}_1 - \mathbf{r}_2\right). \tag{III.9}$$

Now let us observe that quantum hydrodynamics is characterized by the following dimensional quantities: the equilibrium values of ρ and c_s where c_s is the speed of sound (*i.e.* phonons), and the Planck constant $\hbar$. By means of these three quantities it is possible to build up the characteristic scales for the energy E_{QH}, the mass M_{QH}, the length a_{QH}, the frequency ω_{QH} and the energy density ϵ_{QH} within the quantum hydrodynamical context:

$$E_{QH}^4 = \frac{\hbar^3\rho}{c_s}, \qquad M_{QH}^4 = \frac{\hbar^3\rho}{c_s^3}, \qquad a_{QH}^4 = \frac{\hbar}{\rho c_s}, \qquad \omega_{QH} = \left(\frac{c_s^5\rho}{\hbar}\right)^{1/4}, \qquad \epsilon_{QH} \sim \epsilon\left(\rho\right) \sim \rho c_s^2. \tag{III.10}$$

As Landau pointed out[13], the low energy modes present in quantum hydrodynamics are only phonons while the rotational modes (*i.e.* vortices) are separated by a gap. Such a gap is given by the characteristic energy scale E_{QH} above defined. Within the linear regime (and in the absence of rotational degrees of freedom) sound waves are quantized and the phonons obtained have a linear spectrum $E_k = \hbar c_s k$. In the low energy limit the superfluid quantum vacuum behaves as a classical liquid and the quantum fluctuations of the phonon field on the top of this classical background, albeit small, have some influence on its dynamics. Indeed they give rise to the depletion of the mass density ρ of the vacuum which turns out to be an universal phenomenon[6]. This is the well known quantum back-reaction of the vacuum to the phonons which now we quantify.

Indeed the physical picture is the following. At temperature $T > 0$ the liquid is made of vacuum with density ρ and phonon excitations. So it is possible to show how the presence of thermal phonons modifies the mass density of the quantum vacuum and to quantify such a density variation. Let us consider very low temperatures $T \ll E_{QH}$, so that only low-frequency phonons with linear spectrum $\omega = k c_s$ contribute to the thermal energy. Let us also suppose a fixed external pressure. These hypotheses imply that the free energy is the sum of two contributions: the energy of the quantum vacuum and the free energy of the "matter", the phonons. In this case it is possible to show[6] that

$$F(T, \rho) = F_{vac} + F_{mat} = F_{vac} - P_{mat} = \varepsilon(\rho) - \mu\rho - \frac{1}{3}\varepsilon_{mat}(\rho),$$

where $\varepsilon - \mu\rho = \varepsilon_{vac}$ and ε_{mat} are, in this order, the *energy density of the quantum vacuum* and the *energy density of the gas of thermal phonons* (radiation energy). We indicate with ρ_0 the equilibrium density and μ_0 be the chemical potential at $T = 0$; then, since ε_{mat} is considered as a perturbation, we can expand the free energy F in terms of $\delta\rho = \rho - \rho_0$ and $\delta\mu = \mu - \mu_0$ (where $\rho = \rho(T \neq 0)$ and $\mu = \mu(T \neq 0)$). Let us remember that the chemical potential μ must be changed in order to keep a fixed external pressure; furthermore the total change of the pressure of the liquid, which is given by the vacuum pressure of the liquid and the radiation pressure of phonons must be equal to zero.

By taking into account the above considerations and by minimizing over $\delta\rho$ the expression of F just obtained we arrive to the following hydrodynamical equation[6]

$$\frac{\delta\rho}{\rho} = -\frac{\varepsilon_{mat}}{\rho c_s^2}\left(\frac{1}{3} + \frac{\rho}{c_s}\frac{\partial c_s}{\partial \rho} \right), \tag{III.11}$$

which is equivalent to

$$\frac{dc_s}{d\rho} = -\frac{c_s}{3\rho}\left(1 + 3c_s^2\frac{\delta\rho}{\varepsilon_{mat}}\right). \tag{III.12}$$

Now we try to rewrite Eq. (III.12) in a way as similar as possible to Eq. (II.5). By inverting Eq. (III.12), we can write:

$$\frac{d\rho}{dc_s} = -\frac{3\rho}{c_s}\left(1 + 3c_s^2\frac{\delta\rho}{\varepsilon_{mat}}\right)^{-1}. \tag{III.13}$$

Furthermore let us suppose $\frac{\delta\rho}{\varepsilon_{mat}} \ll 1$, *i.e.* we suppose, remembering the definition of $\delta\rho$, that the temperatures we are dealing with will give rise to a very small variation of the density. *As a consequence the temperatures are presumably very close to $T = 0$.* In this way, starting from Eq. (III.13) and by means of a series expansion, we get:

$$\frac{d\rho}{dc_s} \simeq -\frac{3\rho}{c_s}\left(1 - 3c_s^2\frac{\delta\rho}{\varepsilon_{mat}}\right) = -\frac{3}{c_s}\left(\rho - 3c_s^2\frac{\rho\,\delta\rho}{\varepsilon_{mat}}\right) = -\frac{3}{c_s}\left(\rho - c_s^2\frac{\rho\,\delta\rho}{P_{mat}}\right) \doteq -\frac{3}{c_s}\left(\rho + \zeta(\rho)\right), \tag{III.14}$$

with a suitable definition of variables. In this way Equations (II.5) and (III.14) look very similar.

Let us notice that the correspondence between these two equations is

$$c_s \longleftrightarrow a \tag{III.15}$$

and no longer

$$c_s \longleftrightarrow \frac{1}{a^2}, \tag{III.16}$$

as obtained by Barceló et al.[11] upon comparing the spatially flat FRW metric[10] $(ds_{FRW}^2 = -c^2dt^2 + [a(t)]^2 d\overrightarrow{x}^2)$ with the Unruh metric[3] whose components are

$$g_{00} = -\frac{\rho(\mathbf{r},t)}{c_s}(c_s^2 - \mathbf{v}^2), \quad g_{ij} = \frac{\rho(\mathbf{r},t)}{c_s}\delta_{ij}, \quad g_{0i} = -g_{ij}v^j,$$

where $\mathbf{v} = \overrightarrow{v}(t,\overrightarrow{x})$ is the physical velocity of the medium (or superfluid) with respect to the laboratory. In the case of $\mathbf{v} = 0$, then for an inner observer, it is $g_{\mu\nu} = \text{diag}(-1, c_s^{-2}, c_s^{-2}, c_s^{-2})$, the *Minkowskian acoustic metric*[1]. We note explicitly that the result (III.11), from which we deduce (III.14), can be obtained from the analysis of classical hydrodynamic equation made by Stone[14] for which the Unruh metric holds. Nevertheless, the conclusion (III.14) could be derived in an alternative way starting from the effective metric for the superfluid.

We will deal with such a derivation in a forthcoming publication[15].

IV. CONCLUSIONS AND PERSPECTIVES

In conclusion, the arguments given by Volovik[1] in order to justify the deep connections and the analogy between superfluid dynamics and cosmology are here enriched with a new ingredient: the non-zero temperature and the back-reaction of quanta of sound waves onto the quantum vacuum. We been shown that, in this new physical situation, a cosmological-like equation can be derived in a natural way. Let us now remember that measurements of the cosmic microwave background radiation give us the following value for the temperature of our Universe: $T_U = 2.735\,K$, a value very close to $0\,K$. It is interesting to observe how a quantum fluid at low temperature and with a very little variation of the density with respect to the temperature gives rise to an equation formally similar to the fluid equation of an Universe with low temperature and density $\rho = \frac{3H_0^2\Omega_0}{8\pi G} \simeq 9.7 \times 10^{-27} Kg/m^3$, where G is the gravitational constant, H_0 is the Hubble Constant (here supposed equal to its best fit value $= 72\,Km \times s^{-1} \times Mpc^{-1}$), and Ω_0 is the today total density parameter, which for a flat Universe (i. e. our Universe) it is equal to 1.

Finally, concerning Equation (III.14), we argue that it is possible to deduce it in an alternative way from the effective Unruh metric for the superfluid. That will be the subject of a forthcoming publication[15].

V. ACKNOWLEDGEMENTS

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